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The numerical technique for estimation of quench hardness and thickness of hardened case of tool holder after induction brazing and case-hardening in solution of salt and alkali is proposed. The results for cases of case-hardening in quencher at temperature 300, 200 and 100 ºC are presented. It is shown that by variation of the cooling properties of quencher the high hardness of hardened surface can to obtain.

Key words: *computer simulation, case-hardening in solution of salt and alkali, quench hardness, hardened case.*

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CALCULATION OF FUNCTION FOR POPULATION DYNAMICS MODEL WITH CONTINUOUS-TIME AGE

In this article, we present an approach to identify functions in population process model with continuous-time age. The main consideration was given to functions identification of birth, mortality, migration and specific function "becoming mature". Such functions could be defined analytically or by using specific numeric procedure. A numerical example is given to demonstrate the method, along with computer application to population analysis in Canada and Ukraine.

Key words: *population dynamics, the birth rate, functions identification, continuous-time age.*

1. Model description of population dynamics with continuous-time age

Population models are used in biology and ecology to model the dynamics of wildlife or human populations. In common, population is as an aggregate of elements of one biological type subject to particular changes [6]. For social systems, element of population is a person or group of persons (for example, a family, region, county, state, etc). Demographic event means any changes in person status as an element of social system. These changes can cause transfer a particular person from one to another social group. Group usually means aggregate of population elements with similar features (for example, age, income, nationality, etc). Then element transfer from one group to another group defines a process of migration. We can name the group open, if a number of elements could be changed dramatically [7]. In common, we can consider any cells aggregate as population. Some other samples from biological population are cultural plants, wild animals, bacterial cells, virus particles, etc.

Most of publications provide useful theoretical basis for analyzing the sensitivity of stable population distribution and rate of growth to changes in the fundamental parameters [2, 3, 4]. In most cases, features which formed population group are continuous by their nature, for example age, weight and other values (measured by real numbers) [6, 7]. They are interesting from practical and theoretical points of view. In contrast of features described by finite functions, we consider continuous functions for description density of distribution for particular feature.

From a mathematical point of view the model is a linear system of partial differential equations, where the state variables are the population densities in each spatial patch, together with a boundary condition of integral type, the birth equation [1, 5]. A population is divided into $N_p \in N$ groups which correspond to the special characteristics of population (for example, features of welfare, professional features, etc). The main function (state variable) of this model is continuous piecewise differentiate function $n_p(t, \tau)$, which characterises a density of population in time t by age τ on the set of time parameters $(t,\tau) \in V = [0,T][0,\tau_{\max}]$, $p = \overline{1, N_p}$. Then a common number of elements for all groups by age features for particular group p could be defined by the next equation:

$$N_p(t) = \int_0^{\tau_{\max}} n_p(t,\tau) d\tau .$$

Then a model could be described using the next system equations for each group $p = \overline{1, N_p}$:

$$\begin{cases} \dfrac{\partial n_p(t,\tau)}{\partial t} = -l_p(t) \dfrac{\partial n_p(t,\tau)}{\partial \tau} - d_p(t,\tau) h_p(t,\tau), \\ n_p(t,0) = \displaystyle\int_0^{\tau_{\max}} Q(t,s) h_p(t,s) ds, \\ n_p(0,\tau) = \varphi(\tau). \end{cases} \qquad (1)$$

Also consider the main model parameters as functions which characterise population dynamics:

– function of mortality in time t of people in age τ – $d_p(t,\tau)$, as an addition of 2 processes mortality and migration: $d_p(t,\tau) = s_p(t,\tau) + m_p(t,\tau)$;

– function of new beings birth rate for parents in age τ – $Q(t,\tau)$;

– auxiliary function of "becoming mature" speed (motion down the axis of age $\tau \in [0, \tau_{\max}]$) $l_p(t)$

– function of migration speed $m_p(t,\tau)$ by condition of group openness $p = \overline{1, N_p}$.

So, function $n_p(t,\tau)$ is differentiate on the set of $V_d = V / V_Z$, where $V_Z \in V$ – terminal set of transition points, where the system can be changed (functions $l_p(t)$ and $d_p(t,\tau)$ could have breaks). Let's assume $V_Z = \{T_z^0, T_z^1, \ldots, T_z^{N_z}\}$, where $T_z^0 = 0$, and $N_z \geq 0$ – number of breaks on a time set $t \in [0, T]$. In the break points, next conditions should apply:

$$\begin{gathered} l_p\left(T_z^i + 0\right) = c_i l_p\left(T_z^i - 0\right), \\ d_p\left(T_z^i + 0, \tau\right) = z_i d_p\left(T_z^i - 0, \tau\right), \\ c_i, d_i \geq 0 \;\; i = \overline{1, N_z}. \end{gathered} \tag{2}$$

Let's assume $d_p(t,\tau) \equiv d_p(t)$. Then analytical solution of system (1) could be presented like that:

$$\begin{gathered} n_p(t,\tau) = N_p^0 e^{-D_p(t)} F_p^0\left(L_p(t) - \tau\right), \\ D_p(t,s) = \int_0^t d_p(s,\tau) ds, \\ L_p(t) = \int_0^t l_p(s) ds. \end{gathered} \tag{3}$$

Let's consider the next function as a solution (3):

$$\begin{gathered} F_p^0\left(L_p(t) - \tau\right) = \exp\left(-a_p\left(L_p(t) - \tau\right)^2 + b_p\left(L_p(t) - \tau\right)\right), \\ n_p(t,\tau) = N_p^0 e^{-D_p(t,\tau)} \exp\left(-a_p\left(L_p(t) - \tau\right)^2 + b_p\left(L_p(t) - \tau\right)\right). \end{gathered} \tag{4}$$

Then edge condition will be described as:

$$1 = \int_0^{\tau_{\max}} Q(t,s) \exp\left(2a_p L_p(t)s - a_p s^2 - b_p s\right) ds \,. \tag{5}$$

It is possible with the next function of new beings birth rate:

$$Q(t,s)=\begin{cases}0, & s<s_0;\\ Q_0(s-s_0)e^{-q(s-s_0)}, & s\ge s_0.\end{cases} \tag{6}$$

Assume $s_0=16$ – the average age of reproductive period start for human population. If we make substitution (6) to the edge condition (5), we will get the next ratio:

$$q=2a_pL_p(t)-b_p,$$
$$Q_0=\frac{2(a_p+r)\exp(-qs_0)}{\exp\left(-(a_p+r)s_0^2\right)-\exp\left(-(a_p+r)\tau_{\max}^2\right)}. \tag{7}$$

where a_p, b_p, r – constant positive values.

We can study population dynamics trajectories using equations (5-7). In this case, very important role plays numerical and analytical calculations of integral functions $L_p(t)$ and $D_p(t)$.

2. Target setting of model functions identification and its solution algorithm

As we can see from described model above (1-7), functions $d_p(t,\tau)$, $Q(t,\tau)$, $l_p(t)$ and their parameters characterise population dynamics. We will try to calculate these functions in order to identify particular population dynamics. First of all, we should apply the next restrictions:

$$0\le d_p(t,\tau)\le d^{\max}, 0\le l_p(t)\le l^{\max}, 0\le Q_p(t,\tau)\le Q^{\max}.$$

Functions could be define explicitly (for example, as functions (8)), or in a numerical form using defined computational procedure. After such functions definitions, population dynamics, as functions $n_p(t,\tau)$ could be defined on the set of $(t,\tau)\in V=[0,T][0,\tau_{\max}]$, $p=\overline{1,N_p}$ using functions (3,5,7).

But it isn't true for all possible types of model (1). In this case we have to build a difference scheme for definition of mesh functions $x_p(t_i,\tau_j)$ – discrete analog of $n_p(t,\tau)$functions which define on the grid

$$W=\left\{(t_i,\tau_j): i=\overline{1,N}, j=\overline{1,M}; t_i=ih_t; \tau_j=jh_\tau\right\},$$

where $h_t=\dfrac{T}{N}, h_\tau=\dfrac{\tau_{\max}}{M}$. Difference scheme will become as:

$$x_p(t_{i+1},\tau_j)=\left(1-\frac{h_t}{h_\tau}l_i-h_td_i\right)x_p(t_i,\tau_j)+\frac{h_t}{h_\tau}l_ix_p(t_i,\tau_{j-1}), \tag{8}$$

where $l_i=l(t_i)$, $d_i=d(t_i)$, $j=\overline{1,M}$, $i=\overline{0,N-1}$, $p=\overline{1,N_p}$.

Edge and initial condition will be:

$$x_p(t_i,\tau_0)=\sum_{j=0}^{M} h_\tau Q_p(t_i,\tau_j)x_p(t_i,\tau_j),$$

$$x_p(t_0,\tau_j)=\varphi(\tau_j).$$

A scheme (8) will work correctly if:

$$1-\frac{h_t}{h_\tau}l_i-h_t d_i\geq 0 \text{ or } h_t\leq\frac{h_\tau}{l^{\max}+h_\tau d^{\max}}.$$

For analytical solution (5) functions $L_p(t)$ and $D_p(t)$ should be defined as:

$$L_p(t_i)=\sum_{i=0}^{M_i} h_t l_p(t_i),$$

$$D_p(t_i)=\sum_{i=0}^{M_i} h_t d_p(t_i),$$

where $$M_i=\left[\frac{t_i}{h_t}\right].$$

3. A numerical example of parameters definition for Canada-Ukraine demographic model

The data taken from population censuses [8, 9] are among the basic sources of demographic figures which constitute the fundament of a number of analyses. Compared with others, they have the advantage of providing results from the past based on real data. Using this data, we tried to identify functions in models of demographical processes with continuous-time age.

Taking a model described above into consideration, we suggested the next set of model functions (1):

$$l_p(t)=l_0+l_1\sin(w_1 t+w_0);\ d_p(t,\tau)=d_0+d_1\tau+d_2\sin(w_3 t)\tau. \quad (9)$$

Using computer application of such simulation process, we obtained the next parameters value, for Ukraine and Canada respectively:

Table 1

Model parameters value for Ukraine and Canada

№	Parameter	Ukraine	Canada
1	N_0	899	373,6
2	a_p	0,00023	0,00015
3	b_p	0,00027	0,00012
4	l_0	0,726	1,3
5	w_0	0,436	0,00016
6	w_1	0,0506	0,00016

Continuation of table 1

№	Parameter	Ukraine	Canada
7	D	0,31952	0,117
8	d_2	0,00667	0,00003
9	d_1	0,00017	0,00001
10	d_2	0,00002	0,00003
11	w_3	0,6	0,04899
12	T_z^1	2,547	68,75
13	c_1	0,004	0,08
14	d_1	1,525	0,43515

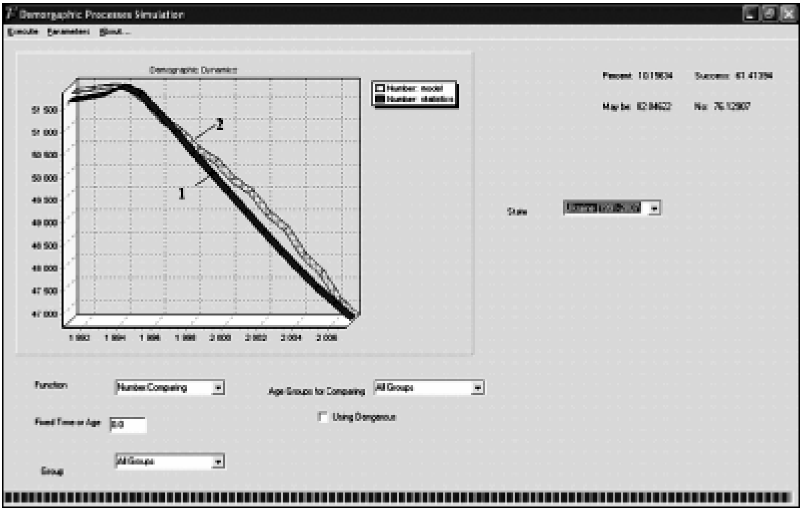

Fig. 1. *Dynamics of birth-rate level in Canada*
(line 1 – statistics, line 2 – experimental)

In **conclusion**, we have to admit one transfer point for Ukraine. Point of break of functions is $T_z^1 = 2{,}547$, which is approximately a year of 1994. Such point of break in Canada was observed at $T_z^1 = 7{,}18$, when function $d_1(t)$ was decreased, probably because of migration wave. Coefficients of population loss functions are considerably small, which explained by compensation mortality by migrations.

Thus, using model (1) and function (2-7) we can investigate qualitative and quantitative characteristics of population dynamics and make comparable analysis on this basis.

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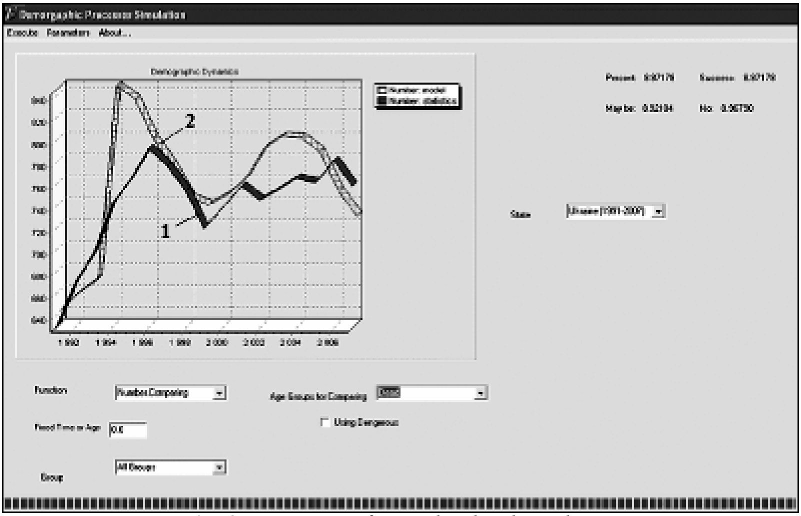

Fig. 2. *Dynamics of mortality level in Ukraine*
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В статті запропонований підхід до визначення функцій – параметрів моделі популяційної динаміки з неперервним віком. Головна увага приділяється ідентифікації функцій народжуваності, смертності та спеціальної функції швидкості дорослішання. Ці функції можуть бути визначенні аналітично, або за допомогою спеціальної чисельної процедури. Для демонстрації моделі розглянутий приклад порівняльного аналізу моделей динаміки населення України та Канади.

Ключові слова: *популяційна динаміка, темп народжуваності, ідентифікація функцій, неперервний вік.*

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