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PRINCIPLES OF PRECISION PARAMETRIC REDUCTION FOR MATHEMATICAL MODELS

Relevancy of the problem of simplification of mathematical models for different classes of objects and phenomena, particularly in the context of dynamic problems while designing computer control systems, is substantiated. Approaches to solving the problem of mathematical models reduction using the concept of coefficient of concordance between characteristics of models and source data accuracy are proposed.

Key words: *mathematical model, reduction, precision, complexity, estimates, concordance coefficient.*

Introduction. Tendency for enhancement of precision in mathematical modeling of objects or phenomena under analysis inevitably leads to escalation of computational complexity of applicable models. Complexity of mathematical descriptions of dynamic objects has a significant impact on possibility of organizing real-time modeling processes for given speed of computer control systems (CCS). In this regard, realizable way to relaxation of CCS performance requirements consists in transition from complex mathematical models to simpler ones, still maintaining requisite information value of the modeling outcome, i.e. preserving adequacy to the object being modeled. Precisely maintenance of adequacy essence of the simplified model and the initial one defines diversity of approaches to simplification of complex models.

The issue of simplification or reduction of mathematical descriptions of the objects or phenomena under analysis is closely associated with the identification problem [1; 2], so that various techniques used for identification can be useful to design a simplified descriptions of dynamic objects for certain complex description as well. Those basic techniques are: decreasing of order of equation system, separation of motion into «fast» and «slow» ones, substitution of operators, reducing the number of independent variables, linearization of initial nonlinear equations, neglect of certain factors influence.

The problem of precision parametric reduction for mathematical models. Solution approaches. The problem of model simplification is formulated as a problem of minimization of certain measure of deviation of outputs of initial and simplified models for a specified simplification pattern. As a rule, it is often assumed that parameters of initial model are known with certainty. Of course, in practice this is not always the case, and therefore the problem can be formulated as concordance of the kind of model and the precision properties of the source data. It is heuristically clear that it makes no sense to use a model that is complex in its structural relationship, if its parameters are known with a sufficiently high inaccuracy. This fact causes a need for coordination of such properties of the models as their implementation accuracy with the source data accuracy. Despite the fact that such setting of model simplification problem can be considered as a certain derivation of the well-known approximate computation problem [3; 4], determination of practical ways to solve it is highly relevant in the study of complex objects and, in particular, in designing of algorithmic and programming support for computer control systems of different purpose.

General and practical enough formulation of mathematical model simplification problem based on the idea of concordance between the model type and source data accuracy was given by A. N. Tikhonov in [5], and it is formulated as minimizing problem of complexity functional for the class of models that are formally comparable in accuracy with the observation (the source data). Class of formally comparable models is determined by inequality constraints for precision measure.

Within the framework of the foregoing general formulation of the problem of mathematical model simplification a number of particular settings can be formulated, one of which is the problem of precision parametric reduction of models.

Suppose we have a mathematical description (model) M of the phenomen on being modeled and the source data (usually parameter values), given with certain inaccuracy, a measure of which is ε_ρ. Initial mathematical description M can be replaced with some other description M' (even much simpler one) with ε_{yp}, measure of inaccuracy of out put variable soft he model, conditioned by inaccuracy of the source data, i.e. presence of ε_ρ. Let $\varepsilon_\rho = 0$, then replacing the original description with any other (even much simpler one) will originate an error $\varepsilon_{ym} \neq 0$. In this case, the question of applicability of the simplified models hould be decided uponexa mining requirements for accuracy of modeling the phenomenon under analysis. In other words, it is permissible to use a simplified model if $\varepsilon_{yM} \leq \varepsilon_з$, where $\varepsilon_з$ is a preassigned value of modeling error measure, and vice-versa. If there is a fair lystrongine quality $\varepsilon_{yM} < \varepsilon_з$, it implies the possibility of further simplification of the initial model.

In general case, applicability of mathematical model M or M' is determined by the condition

$$\varepsilon_{yM} + \varepsilon_{yp} \leq \varepsilon_з . \tag{1}$$

Here ε_{yM} is either a measure of error of description of the phenomenon under analysis by the model M, or measure of error conditioned by replacement of given model M with any other model M' (even much simpler one). Since in this study we consider simplification of mathematical models, hereinafter ε_{yM} will stand for measure of error conditioned by replacement of the initially given model with a simpler one.

The following value is taken as a characteristic property of concordance of the source data error with the mathematical model M' that replaces the model M

$$\alpha = \varepsilon_{yM} / \varepsilon_{yp} , \tag{2}$$

we'll call it concordance coefficient. Thus condition of applicability (1) of the model M' takes the following form

$$(\alpha + 1)\varepsilon_{yp} \le \varepsilon_3 . \tag{3}$$

Definition 1. Mathematical model M' is called α -concordant with the source data error ε_p of the model M, if $\varepsilon_{yM} = \alpha\varepsilon_{yp}$ and $\alpha > 0$.

The problem of concordance of the model M with degree of precision of the source data means obtaining a model M' with a certain value of concordance coefficient. As a lower bound may be taken, for instance, the value $\alpha = 0.01$, while further concordance coefficient decrease, which means the use of models that differ slightly from the initial one, will not lead to a significant gain in accuracy of modeling. On the other hand, if $\alpha > 10$, the overall modeling error will be determined almost entirely by ε_{yM}, in which case while modeling complicated objects, satisfactory values of ε_{yM} are practically impossible. Thus, the concordance coefficient of mathematical models should be substantially limited by the interval of (0.01 $\div$ 10).

While solving specific problems, maximum value of concordance coefficient $\alpha_{\max}$ is determined by the condition of applicability (3) as follows:

$$\alpha_{\max} = \varepsilon_3 / \varepsilon_{yp} - 1 . \tag{4}$$

Now let N_M be a measure of computational complexity of mathematical model M (for dynamic objects, such as described by ordinary differential equations in standard form, a number of operations reduced to the same type and needed to compute the right-hand part may be taken as N_M), and μ_α is a set of mathematical models, coefficient of concordance of which with source data error does not exceed α.

If $\alpha < \alpha_{\max}$, then all models from μ_α are equivalent in terms of applicability, since for any model the condition of applicability (3) is met. However, models from μ_α may not be equivalent in terms of computational complexity. Exactly under this condition possibility appear to simplify the given model M by replacing it with a model $M' \in \mu_\alpha$ for which $N_{M'} < N_M$.

Definition 2. Mathematical model M'_0 is called optimally simplified if

$$N_{M'_0} = \min_{M' \in \mu_\alpha} N_{M'} \tag{5}$$

It is obvious, that measure of computational complexity $N_{M'}$ is a decreasing function of concordance coefficient. Consequently, concordance coefficient of optimally simplified model M'_0 exceeds α .

Building α -concordant and optimally simplified models has a twofold purpose. On the one hand it is justification of certain assumptions that permit to simplify the model, on the other hand it is relaxation in the requirements for performance of computer control system modeling systems. Yet the practical solution of the problem under concern faces considerable difficulties, due mainly to complexity of the problem of estimation of values ε_{yM} , ε_{yp} , as well as difficulties in formalization of the set μ_α .

Practical way of coping with these difficulties consists in the use of parametric approach to construction of simplified models. Its essence is to replace the values of certain parameters of the initial model M with such values, which reduce the value of computational complexity measure. Usually given values of certain parameters of the initial model are replaced with zero values (both for additive and multiplicative parameters) and/or with unit values (for multiplicative parameters). Parametric simplificationof this kind is worth using if mathematical models of processes or phenomena under analysis contain sufficiently large number of parameters, values of which are determined from experimental data. These models include models in the form of systems of linear and nonlinear algebraic equations of higher dimension, systems of differential equations (both in ordinary and partial derivatives) with coefficients that are composite functions of input and output variables, integral equations with kernels, form of which is determined by objective parameters of objects under analysis etc.

Theoretical justification of possibility of parametric simplification of mathematical models for building of α -concordant and optimally simplified models is based on the results of the theory of accuracy as applied to inherent error analysis.

Let $M(y, V, \rho)$ be mathematical model; y, V, ρ — vectors of output, input variables and parameters, respectively, that belong to normed spaces with metrics ρ_y, ρ_V, ρ_p, defined by the relevant norms; N_p is a measure of computational complexity.

Let ρ_0 be an unknown vector of precise values of parameters of the model, and ρ_3 be a vector of preset values of the parameters. Besides it is known that

$$\rho_p(\rho_3, \rho_0) \le \varepsilon_p . \quad (6)$$

Condition (6) defines such a set Ω_p, that from $\rho \in \Omega_p$ results $\rho_p(\rho_3, \rho) \le \varepsilon_p$. Thus a set μ of possible models is defined

$$\mu = \left\{ M(y, V, \rho) : \rho \in \Omega\rho, \rho_p(\rho_3, \rho) \le \varepsilon_p \right\},$$

and the model $M(y, V, \rho)$ is an initial mathematical description.

Suppose further that $\rho_y(\rho_3, \rho)$ is a measure of error in the space of output variables, that is conditioned by inaccuracy of determination of the vector ρ_3. $\rho_y(\rho_3, \rho)$ is a measure of error, that is conditioned by replacement of the initial model $M(y, V, \rho_3)$ with another model, in particular a simpler model $M(y, V, \rho_r)$; $\rho_y(\rho_3, \rho)$ is a measure of error that characterizes distinctive feature of the accurate model $M(y, V, \rho_0)$ from the model $M(y, V, \rho_r)$.

Estimates can be determined for the errors referred to above

$$\rho_y(\rho_3, \rho) / \rho \in \Omega\rho, V \in \Omega\rho \le \varepsilon_{yp}, \quad (7)$$

$$\rho_y(\rho_3, \rho_r) / V \in \Omega v \le \varepsilon_{yM}, \quad (8)$$

$$\rho_y(\rho_0, \rho_r) / V \in \Omega v \le \varepsilon_{yr}. \quad (9)$$

(Ωv — range of possible values of the input variables), and the triangle inequality defines relationship between them

$$\varepsilon_{yr} \le \varepsilon_{yM} + \varepsilon_{yp} = (\alpha + 1)\varepsilon_{yp} . \quad (10)$$

If $\rho_r \in \Omega\rho$, then $\varepsilon_{yM} \le \varepsilon_{yr}$, and consequently, the set μ of possible models represents a set of α -concordant models if α =1, i.e. $\mu = \mu_1$. However, application of parametric approach to the search for simplified models in the set μ has a limited scope. Indeed, a set $\Omega\rho$ usually does not contain points that belong to the coordinate axes, and therefore with such simplification of models one can rely only on reduction of the number of significant digits in representation of the parameters specified. It should be noted that such a simplification is applicable in approximate computation technique and can be used also in organization of computation processes in modeling systems on the stage of selection of algorithms, programs and specialized computation tools.

In general case, if $\rho_r \in \Omega\rho$, then $\varepsilon_{yM} \le \varepsilon_{yr}$, and consequently concordance coefficient of the respective model will be greater than one. Nevertheless, there is a theoretical possibility of determining the set of α -concordant models with $\alpha \le 1$ and $\rho_r \overline{\in} \Omega\rho$. Existence of this possibility is conditioned by the fact that error in the space of output variables, caused by inaccuracy of definition of the vector ρ_3 is measured in the metric ρ_y.

Indeterminacy of values of the initial model parameters vector (presence of $\Omega\rho$) leads to indeterminacy in values of the output variables, quantitative measure of which is the error

$$\Delta y = y(\rho, V) - y(\rho_3, V) / V \in \Omega_V . \qquad (11)$$

The range $\Omega_{\Delta y}$ of possible values of the error Δy has a complicated configuration in the space of output variables, that depends on the form of Ω_ρ, properties of mathematical description of $M(y, V, \rho)$, and is determined by inherent error equation. Detailed description of $\Omega_{\Delta y}$ by inherent error analysis techniques, developed in the theory of accuracy, can be obtained for the simplest problems only. However usually the membership range of an error is characterized by some sphere S (in a predetermined metric ρy), radius of which (in this case ε_{yp}) is defined on the basis of upper bound (7) for the norm that corresponds to the metric ρy. Due to coarseness of the estimates applied, it turns out that $S \supset \Omega\Delta y$. In the space of output variables, certain range $\Omega'\rho$ in the parameter space corresponds to the sphere S, and from $\Omega'\rho \supset \Omega\rho$ and $\rho \in \Omega'\rho$ follows $\rho y(\rho_3, \rho) \le \varepsilon_{yp}$. Therefore if $\rho_r \in \Omega'\rho$, then $\varepsilon_{yM} \le \varepsilon_{yp}$, and the set μ' of possible models, that corresponds to the set $\Omega'\rho$, is a set of consistent models for α =1.

Due to the presence of inclusion $\Omega \subset \Omega'\rho$, application of parametric approach to simplification of the initial description in the class of models μ' has more potentials as compared to the class μ. It is because for complicated problems with a large number of parameters, specified with not very high accuracy, the range $\Omega'\rho$ contains the points, that belong to the coordinate axes of the parameter space. Consequently, vector with some zero-components can be selected as a vector ρ_r, and thus reduction of the computational complexity measure value N_{pr} is achieved.

In general case, $\Omega'\rho$ may not intersect with the coordinate axes, but even under these conditions, as it is easy to show, there is a set $\Omega'\rho(\alpha) = \left\{\rho : \rho_y(\rho_3, \rho) \le \alpha\varepsilon_{yp}, \alpha > 1\right\}$, that includes points of the coordinate axes and defines the set of models μ'_α, concordant with the source data error with concordance coefficient $\alpha > 1$. Indeed, let $\Omega'\rho(\beta) = \left\{\rho : \rho_p(\rho_3, \rho) \le \rho\varepsilon_p\right\}$. It is clear, that $\Omega_p(1) = \Omega_p$, and there is such a value $\beta_0 = 1$, that if $\beta > \beta_0$, the sphere $\Omega_p(\beta)$ is intersected by at least one of the coordinate axes of the parameter space. The set $\Omega_p(\beta)$ defines a range of possible output variables error values $\Omega_{\Delta y}(\beta)$, which is estimated in the metric ρy by the sphere S_α with radius $\alpha(\beta)\varepsilon_{yp}$, i.e.,

$$\rho_y(\rho_3, \rho) / \rho \in \Omega'\rho(\beta) \le \alpha(\beta)\varepsilon_{yp} . \qquad (12)$$

Here $\alpha(\beta)$ is a nonnegative nondecreasing function, and $\alpha(1) = 1$, we'll assume that $\alpha(\beta)$ is an increasing and bounded function. The first assumption means, that if source data error increases, the output variables error increases also (for practical problems it can be assumed that this condition is satisfied). The second condition corresponds to the correctness property of all the models with parameters $\Omega_p(\beta)$, that may constitute a limit of parametric approach applicability to simplification of models.

Suppose now that $\Omega'\rho(\alpha) = \{\rho : \rho_y(\rho_3, \rho) \le \alpha(\beta)\varepsilon_{yp}\}$ is a prototype in the space of parameters of the sphere S_α . Since $\Omega'(\alpha) \supset \Omega(\beta)$, then the set $\Omega'\rho(\alpha)$ also intersects with at least one of the coordinate axes, which was to be demonstrated. Thus we succeeded in proving the following theorem.

Theorem 1. For any $\Omega\rho = \{\rho : \rho_p(\rho_3, \rho) \le \varepsilon_p, \varepsilon_p > 0\}$ there is such $\alpha_0 > 0$, that if $\alpha > \alpha_0$, the set μ_α of correct models $M(y, V, \rho)$, α -concordant with the source data error, defined by a set of parameters $\Omega\rho(\alpha) = \{\rho : \rho_y(\rho_3, \rho) \le \alpha\varepsilon_{yp}\}$, includes a non-empty subset $\bar{\mu}_\alpha$ of models $M(y, V, \bar{\rho})$, for which $N_{\bar{p}} < N_p / \rho \in \Omega\rho$.

This theorem is a theoretical justification for applicability of parametric approach to construction of simplified mathematical models based on concordance of the type of model with the source data accuracy. And the fact, proved by the theorem, of existence of α -concordant models with a lower measure of computational complexity in comparison to the initial model, is the essence of the principle of precision parametric reduction of mathematical models, that can be formulated as follows.

For any model with imprecisely defined parameters, provided that the output variables error is an increasing and bounded function of the parameter error, there is a simpler α -concordant model, that differs from the initial one by values of certain parameters and reduces computational complexity measure.

Of course, applicability of parametrically simplified models resulting from precision parameter reduction is defined by the condition (3), so creation of mathematical descriptions, that would be simpler than the initial ones and would have not very large concordance coefficient, is of practical importance. It is therefore desirable to be able to assess concordance coefficient of parametrically simplified models a priori and with relatively simple technique.

Evaluation of inherent error within the framework of linear accuracy theory for a wide class of problems (mathematical descriptions of which represent models that can be simplified) leads to a linear dependence of ε_{yp} from ε_p [6], i.e.,

$$\varepsilon_{yp} = C_M \varepsilon_p , \tag{13}$$

where C_M is a constant determined by the properties of the problem (for correct problems it is the norm of inverse operator of inherent error linear equations).

In this case, it is not difficult to get a guaranteed upper bound for the value α_0 of concordance coefficient of the models, that can be obtained as a result of precision parametric reduction.

Theorem 2. For the minimum value α_0 of concordance coefficient of parametrically simplified model $M(y,V,\overline{\rho})$ the following estimate is valid

$$\alpha_0 \le \frac{\min\limits_{i=\overline{1,m}} \rho_p(\rho_3, \wp i)}{\varepsilon\rho} = \alpha_b , \tag{14}$$

where $\wp i, i=\overline{1,m}$ are the coordinate axis in the parameter space.

Proof. Condition (13) implies that the function $\alpha(\beta)$ has the form $\alpha(\beta)=\beta$. Value β_0 will be determined from the condition that the sphere $\Omega\rho(\beta)$ with radius $\beta_0\varepsilon_p$ touches the nearest of the coordinate axes of the parameter space $\wp i, i=\overline{1,m}$, i.e.

$$\beta_0 \le \frac{\min\limits_{i=\overline{1,m}} \rho_p(\rho_3, \wp i)}{\varepsilon\rho} .$$

From the inclusion $\Omega'(\alpha(\beta_0)) \supset \Omega(\beta_0)$ results $\alpha_0 \le \beta_0$. The theorem is proved.

The estimate (14) is substantially an applicability criterion of precision parametric reduction of mathematical models. Thus, higher values of α_b (about tens) denote inexpediency of parametric simplification of models and on the contrary, lower values of α_b (about units) evidence admissibility of precision parametric reduction for construction of suitable simplified models.

The parametric simplification of models under consideration also covers the case with linearization of complex functional relationships.

Meanwhile the use of linear terms means «zeroing» of coefficients for all other expansion terms (approximation of nonlinear relationship).

Furthermore, application of the principle of precision parametric reduction is possible also in order to assess insignificance of separate fragments of the model by introducing fictitious multiplicative parameters (nominally equal to one) for the fragments under consideration with subsequent evaluation of insignificance of these parameters, i.e. with assessment of the possibility of replacing their nominal values with zero.

Conclusions. In summary, a number of theoretical points is outlined, substantiating availability of constructive approaches to solving the problem of precision reduction of mathematical models. The arguments adduced allow us to determine expediency and practicability of selection or developing of reduced description of the modeling problem to be solved, with the help of estimation of parameter of concordance between the initial model and the simplified one.

References:

1. Prochazka A. Signal Analysis and Prediction / A. Prochazka, N. Kingsbury, P. J. W. Payner, J. Uhlir. — Boston : SpringerScience + Business Media New York, 1998.
2. Ivahnenko A. G. Inductive method of self-organizing models of complex systems / A. G. Ivahnenko. — K. : Naukova Dumka, 1982.
3. BahvalovN. S. Modern Problems of Computational Mathematic sand Mathematical Modeling / N. S. Bahvalov, V. V. Voevodin. — M. : Nauka, 2005.
4. Shevchuk B. M. Technology of multifunction processing and transfer of information in the monitoring networks / B. M. Shevchuk, V. K. Zadiraka, L. O. Gnativ, S. V. Fraer. — Kyiv : Naukova Dumka, 2010.
5. TihonovA. N. Mathematical methods of automating the processing of observations / A. N. Tihonov // Problems of Computational Mathematics. Moscow University Press. — 1980. — P. 3–17.
6. Cobzas S. Functional Analysis in Asymmetric Normed Spaces / S. Cobzas. — Basel : Springer Basel, 2013.

Обґрунтовується актуальність завдання спрощення математичних моделей різного класу явищ і об'єктів, втому числі стосовно завдань динаміки при розробці обчислювальних керуючих систем. Пропонуються підходи до вирішення завдання редукції математичних моделей, засновані на понятті коефіцієнта узгодженості характеристик моделей і точності вихідних даних.

Ключові слова: *математична модель, редукція, точність, складність, оцінки, коефіцієнт узгодження.*

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